Nonregular Languages

Lecture 15

Section 4.3

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Outline

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Nonregular Languages

- It turns out that many languages cannot be recognized by DFAs.
- For example,

$$L = \{w \mid w \text{ has an equal number of } \mathbf{a}\text{'s and } \mathbf{b}\text{'s}\}$$

  is not regular.
- How can we prove that?

Outline

The Pumping Lemma (for Regular Languages)

Theorem (The Pumping Lemma)

If L is an infinite regular language, then there exists a positive integer m such that, for every string $w \in L$ of length at least m, w can be decomposed as $w = xyz$ such that

- $|xy| \leq m$,
- $|y| \geq 1$,
- *For every $i \geq 0$, $xy^iz \in L$,*

- We will call m the pumping length of the language.

Theorem (The Pumping Lemma)

For every regular language L, there exists a positive integer m such that, for every string $w \in L$ of length at least m,there exist strings x, y, and z such that

- $|xy| \leq m$,
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The Pumping Lemma (for Regular Languages)

Theorem (The Pumping Lemma)

For every regular language L, there exists a positive integer m such that, for every string $w \in L$ of length at least m, there exist strings x, y, and z such that $w = xyz$ and

- $|xy| \leq m$,
- $|y| \geq 1$,
- *For every* $i \geq 0$, $xy^iz \in L$,

The Pumping Lemma

Proof, beginning.

- Let L be a regular language.
- Let M be a DFA that recognizes L.
- Choose M to be the number of states in M.
- Let $w \in L$ be a string of length $\ell \geq m$.
- When M processes w, it begins in state q_0 and proceeds to a new state for each symbol in w.

□

The Pumping Lemma

Proof, continued.

- Label the visited states $s_0, s_1, s_2, \ldots, s_\ell$, where $s_0 = q_0$.
- This list contains $\ell + 1 > m$ states.
- Therefore, one state must be repeated.
- Let s_j be the first state repeated and let s_k be the first repetition of s_j. That is, $s_j = s_k$ and $k > j$.

□

Proof, continued.

- Let x be the string of symbols processed in getting from s_0 to s_j,
- Let y be the string processed in getting from s_j to s_k, and
- Let z be the string processed in getting from s_k to s_ℓ.

□

The Pumping Lemma

Proof, concluded.

- Then, clearly,
  - $s = xyz$,
  - $|xy| \leq m$.
  - $|y| \geq 1$,
- It follows that $xy^iz \in L$ for any $i \geq 0$ because we can follow the loop from s_j back to s_j (i.e., s_k) as many times as we like, including not at all.

□

Outline

Examples

Example (Nonregular languages)

- Show that the language $L = \{\mathbf{a}^n\mathbf{b}^n \mid n \geq 0\}$ is nonregular.

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Theorem (The Pumping Lemma)

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- $|xy| \leq m$,
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- Let $L = \{\mathbf{a}^n\mathbf{b}^n \mid n \geq 0\}$ and suppose that L is regular. (Your choice of L)
- Let m be the "pumping length" of L. (Your worst enemy's choice of m)

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- Let $w = \mathbf{a}^m\mathbf{b}^m \in L$. (Your choice of w)
- Then $w = xyz$ where $|y| \geq 1$ and $|xy| < m$. (Your worst enemy's choice of x, y, and z)

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- According to the Pumping Lemma, $xy^2z = \mathbf{a}^{m+k}\mathbf{b}^m \in L$, which is a contradiction.

Examples

Example (Nonregular languages)

- Let $L = \{\mathbf{a}^n\mathbf{b}^n \mid n \geq 0\}$ and suppose that L is regular. (Your choice of L)
- Let m be the "pumping length" of L. (Your worst enemy's choice of m)
- Let $w = \mathbf{a}^m\mathbf{b}^m \in L$. (Your choice of w)
- Then $w = xyz$ where $|y| \geq 1$ and $|xy| < m$. (Your worst enemy's choice of x, y, and z)
- It follows that $y = \mathbf{a}^k$ for some $k > 0$. (Your choice of k)
- According to the Pumping Lemma, $xy^2z = \mathbf{a}^{m+k}\mathbf{b}^m \in L$, which is a contradiction.
- Therefore, L is not regular.

Examples

Example (Nonregular languages)

- Show that the following languages are nonregular.
  - $\{ww^R \mid w \in \Sigma^*\}$.
  - $\{w \mid w$ has an equal number of **a**'s and **b**'s$\}$.
  - $\{w \mid w$ has an unequal number of **a**'s and **b**'s$\}$.

Examples

Example (Nonregular languages)

- Show that the language L of all correct multiplication problems is non-regular.
- For example, the problem $\begin{array}{r} 00101 \\ 00110 \\ \hline 11110 \end{array}$ would be represented by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

- Consider the multiplication

$$(2^n - 1) \times (2^n - 1) = 2^n(2^n - 1) - 2^n + 1$$

for $n \geq 1$.

Collected

To be collected on Fri, Sep 30:

- Section 3.1 Exercises 21b, 22.
- Section 3.2 Exercises 15a.
- Section 3.3 Exercises 11, 12.
- Section 4.1 Exercises 1a, 16 (give proof).

Outline

Assignment

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- Section 4.3 Exercises 1, 3, 4, 5bdef, 8, 18aef, 19, 20, 24, 26.